Avengers College

UNIVERSITY OF CAMBRIDGE
Computer Laboratory

Computer Science Admissions Test

EXAMPLE QUESTIONS. *These are meant as practice material and have various degrees of difficulty. Some are inspired from past CSAT papers.*

Instructions:

- The test duration is 100 minutes. Section A has 8 questions. Section B has 12 questions which are more challenging and worth more marks.
- All questions attempted are marked. Your best 5 questions from each section are considered. Partial answers are taken into account. You can choose the questions to answer and their order.
- Write only on the work-booklet provided and clearly label the question you are solving at the top of each page. Answers without working may not gain full marks. You should show sufficient working to make your solutions clear to the Examiner, but these need not be extremely thorough.
- Calculators, phones, watches, smart-glasses or other electronic devices or paper are **not** permitted.
- All paper must be handed in. Do not write on the cover or question sheets.
- **Do not** discuss any test questions with others (e.g. candidates at the same or another College, the Internet, or elsewhere), especially before March. You would disadvantage yourself.

It is recommended that you:

- take 5 minutes first to read through all questions,
- start with Section A and spend no more than 30 minutes on it,
- aim for 5 questions in each section; if you finish early then attempt more from Section B.

Good luck!

Section A — aim for 5 questions out of 8, of your choosing

1. Sketch the function $f(x) = \min_{t<x} t^2$ for all real x.

2. Five candidates took a maths test and got scores A, B, C, D and E, with $A > B$, $C > D$, $D > B$ and $E > B$. In how many possible ways could the candidates be ranked?

3. Find positive integers a, b, c, d such that $a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}} = \frac{15}{11}$.

4. Three planar regions A, B, C partially overlap each other, with $|A| = 90$, $|B| = 90$, $|C| = 60$ and $|A \cup B \cup C| = 100$, where $|\cdot|$ denotes the area. Find the minimum possible $|A \cap B \cap C|$.

5. Find the gradient of the implicitly defined curve $y^2 + 2y = x^3 + 7x$ at all its intersection points with the line $x = 1$.

6. What does $\lim_{x\to\infty} \frac{f(x)}{f(-x)} = -1$ imply about a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with real coefficients? Prove your answer.

7. A ternary tree is a collection of nodes, each having 0, 1, 2 or 3 children, where one node is not the child of any node and each of the other nodes is a child of exactly one other node. What are the maximum and minimum possible depths of a ternary tree containing n nodes?

8. A polynomial $f(x) = x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ with integer coefficients a_i has roots at 1, 2, 4, $\ldots$, 2^{n-1}. What possible values can $f(0)$ take?

Section B — aim for 5 questions out of 12, of your choosing

9. Using a fair coin we can generate random integers in $\{1, 2, 3, 4\}$ with equal probability by doing:

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(a) toss coin, if heads go to (b) otherwise go to (c)
(b) toss coin, if heads output 1 otherwise output 2
(c) toss coin, if heads output 3 otherwise output 4
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By altering just one of the lines (a), (b) or (c), we can generate random integers in $\{1, 2, 3\}$ with equal probability. Identify which line and give the new version. Prove that it is correct.

10. A line $y_1 = ax + b$ is tangent to the curve $y_2 = 12 - x^2$, with $0 < x < \sqrt{12}$. Find the reals a, b such that the area delimited by y_1, the x-axis and the y-axis is minimized.

11. You must remove the k smallest elements from a list of length n. Strategy (1) sorts the list in ascending order, which takes time n^2, then removes the first k elements. Strategy (2) repeatedly finds the smallest element by scanning the list and removing it until k elements are removed. The time taken to compare two elements is 1 and the time taken to remove an element is 1. All other operations take zero time. Prove that strategy (2) is faster.

12. A point traces a unit circle if its coordinates satisfy $(x, y) = (\cos t, \sin t)$ as time t varies from 0 to 2π. Give an equation for a point that traces a spiral centred at $(0, 0)$ and that crosses the positive x-axis at $x = 1, 2, 3, \ldots$ at times $t = 2\pi, 4\pi, 6\pi, \ldots$, and find its speed $v(t)$ at time t.

13. You place right-angled triangles ABC with $\angle A = 360/k$ degrees, where $k \geq 5$ is an integer and $\angle B = 90$ degrees, as follows: the first triangle with A at $(0, 0)$ and B at $(0, 1)$, then repeatedly place triangles with A at $(0, 0)$ such that AB of the new triangle coincides with AC of the previous triangle. What is the size of the area enclosed by the first k triangles?

14. Find all positive integers n such that $n + 3$ divides $n^2 + 27$.

15. A bag has 3 black and 3 white balls. A game starts with you extracting 3 balls at random. If they are the same colour you win. Otherwise you return the two balls with the same colour to the bag, discard the other ball, and start again removing 3 more balls at random. If in the end the bag contains 2 balls you lose. What is the probability of winning the game?

16. The eight numbers $11, \ldots, 18$ are in a database in some order. You can query any subset of indices, but the reply will be randomly shuffled. For example, if the order was $17, 12, 13, 16, 11, 15, 14, 18$ and you queried indices $1, 2, 4$, the reply could be $16, 17, 12$. What is the minimum number of queries you must make to determine the order of the eight numbers?

17. You have n balls numbered $1, 2, \ldots, n$ that you are placing in n buckets. In how many ways can you do this such that: *(i)* no bucket remains empty? *(ii)* exactly one bucket remains empty?

18. How many squares (including tilted ones) can be built with vertices on a grid of $n \times n$ points? The following may be useful: $\sum_{k=1}^{n} k^2 = n(n + 1)(2n + 1)/6$ and $\sum_{k=1}^{n} k^3 = n^2(n + 1)^2/4$.

19. A population $x(t)$ grows in time according to $dx/dt = (x - 1)(2x - 1)$. Knowing that $x(0) = 0$, after how much time does it reach reach 50% of its ultimate value as time passes?

20. Show that $2^{50} < 3^{33}$.

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